RESEARCH ISSUES IN UNDERGRADUATE MATHEMATICS LEARNING

PRELIMINARY ANALYSES AND RESULTS

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THE ROLE OF EMOTION: EXPERT AND NOVICE MATHEMATICAL PROBLEM-SOLVING

A Preliminary Study

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In *A Call for Change: Recommendation for the Mathematical Preparation of Teachers of Mathematics* (Leitzel, 1990), the Mathematical Association of America joins the National Council of Teachers of Mathematics (1989, 1991), the Mathematical Sciences Education Board (1990), Sigma XI (1989), and other professional organizations (Kenschaft, 1990) in including the role of affect or emotion as a construct deserving of the attention of teachers and researchers. Of concern is how a positive attitude toward mathematics can be developed, what techniques can be employed to foster the joy of learning, and how to develop an environment that is perceived as encouraging rather than hostile or overly snobbish and competitive (Buerk, 1982, 1988; Rosamond, 1984). An additional goal is to find ways to involve students more actively in their own learning.

During the last 15 years, considerable progress has been made in understanding the importance of affective variables in relation to mathematics-learning. The initial motivation for most of the work, begun in the mid–seventies, was the desire to understand variables related to gender differences (Fox *et al.*, 1977). By the mid–eighties, isolated mathematicians throughout the world were convinced emotion was a powerful factor in mathematics-learning for all students. In 1985 and 1986, the Canadian Mathematics Education Study Group (CMESG) invited Poland and Rosamond to offer Study Groups to explore the role of feelings in learning mathematics and techniques in organizing classrooms (Rosamond, 1985). In 1986 and 1987 in San Diego, McLeod (1989) arranged a conference to discuss mathematics problem–solving in relation to the theoretical framework of emotion developed by George Mandler (1975).

These discussions revealed that beliefs, attitudes, and values are elements of what has generally been called the affective domain and that they each have an impact on cognition. Mandler prefers to use the term "emotion" to refer to the intense "hot" passions, such as fear, joy, or anger, and which require some kind of visceral arousal and which may be accessed very quickly (Mandler, 1989). The theories of Mandler, Lazarus (1980, 1986), and other information-processing constructivists are attractive to educators because they admit mediating or coping mechanisms that allow evaluation of emotion and cognitive realignment with one's goal. These theories suggest that emotions can be predicted and used to improve mathematics learning (perhaps) as part of what Skemp defined as reflective intelligence: the ability to make one's own mental processes the object of conscious observation and to change these intentionally from a present state to a goal state.

In an attempt to begin to describe the types of emotional reactions that occur during mathematical problem–solving and to identify ways in which emotion inhibits or enhances learning, a two–part research project was designed in which mathematics experts and then novices engaged in observer/solver, paired problem–solving, exercises. This chapter will describe the exercises and results, make some comparisons between the experts and novices, and render some claims about the value of the exercises.

POSITIVE EMOTIONS AND MATHEMATICS PROBLEM–SOLVING

It was hoped that positive emotions that contributed to the problem–solving process would be observed. Positive emotions tend to be frowned upon or viewed as "childish." Few people exhort optimism like Ray Bradbury: "We are matter and force turning into imagination and will! I am the center of a miracle! Out of the things I am crazy about I've made a life. ... Be proud of what you're in love with. Be proud of what you're passionate about!" (Bradbury, 1982). People who exhibit positive emotions often are regarded as playing, as not being serious.

Yet, playing with ideas is inherent in mathematics problem–solving. Then what emotions should we expect to feel? Lazarus (1986) answers this by saying that the essence of play is that it is highly stimulating.

It is accompanied by pleasurable emotions such as joy, a sense of thrill, curiosity, surprise, wonder — emotions exploratory in nature. Stephen Brown (1971, 1982) is one mathematics educator who has explored ways of promoting those emotions in teaching.

As educators we are concerned about the classroom environment. Are there optimum conditions for encouraging problem–solving activity? "Exploratory activity occurs more readily in a biologically sated, comfortable and secure animal than in one greatly aroused by a homeostatic crisis," observed Lazarus, "The human infant will not venture far from a parent unless it is feeling secure, at which point it will play and explore, venturing farther and farther away but returning speedily if threatened or called by the mother." The novices in our study felt increasingly secure due to the support of their partners, and reported more out–of–class mathematical discussion.

The identification of emotion was through self–report and peer-observation, with no external verification of the visceral component (e.g., measurement of change of temperature or blood pressure, for example) or measurement of intensity or duration. Observers were, however, asked to record sighs, laughter, twitches, wiggles, tapping, yawns, or flushes. A heightened emotional intensity appeared to occur at four stages of the problem–solving process: (1) at the initial reading of the problem, (2) when embarking on the problem–solving process, (3) during immersion in the problem, and (4) when a solution was found or when the session was about to end.

Additionally, there appeared to be a rhythm or cycle during the engagement with the problem: first, attention to the problem or "immersion," then outward attention or "distraction," followed by total immersion, then distraction, etc. Integrated with, and slowing down, the cycle were moments when the attention remained on the problem but the accompanying actions were mere "holding patterns," such as rewriting the question, with almost no cognitive load. These appeared to be times of incubation, cognitive "breathers," rather than times of woolgathering. No stress was involved, and all attention remained on the problem. The next section of this chapter outlines the research procedure for the experts and is followed by a discussion of those results.

EMOTION IN EXPERTS' PROBLEM–SOLVING

The data discussed in this section were obtained over a three–year period in the United States and Canada. In the United States, 12 subjects were professional mathematics educators or were employed in business or technical fields in which they were accustomed to using mathematics. Another six subjects were graduate students in a mathematics problem–solving course at a state university. In Canada, 19 subjects were mathematics educators participating in study groups on the role of feelings in learning mathematics held during two annual meetings of the Canadian Mathematics Education Study Group (CMESG). Except for the graduate students, the participants were all seasoned mathematicians age 40 or older, eight of whom were women. Data were not gathered on the mathematicians' specialty, e.g., topology, algebra.

PROCEDURE FOR EXPERT'S PROBLEM SOLVING

The main activity was paired–problem solving, where one member of the pair worked on a mathematical problem under the observation of the partner. We intended to make as transparent as possible the emotions and cognitive activity during the process. The format of each occasion was the same. Six to 10 subjects gathered together with two co–leaders (John Poland, Carleton University and Rosamond at CMESG; Maria Arrigo, San Diego State University and Rosamond in San Diego), engaged in a general discussion of emotion and then read a list of words and phrases that might indicate various emotions. Our emotions seemed to have more shades of meaning than we had names for. We discussed physical manifestations of feelings, such as laughter that signifies nervousness or yawning that signifies confusion. Subjects suggested that some visceral sensations, such as butterflies in the stomach, could be associated with either positive anticipation or dread.

The participants then paired–up, one person agreeing to be the problem solver while the other was the observer. The problem solvers then browsed through books and handouts supplied by the leaders to choose a problem. Problems were nontrivial but within the knowledge range of the participants. They were chosen from puzzle books by Martin Gardner (1967, 1979) and Mott–Smith (1954) and from books by Honsberger (1970, 1973).

The problem solvers were to work as transparently as possible and to provide a running commentary on their thoughts and emotions. The observers were to pay attention and take notes, to prompt the subject to verbalize whenever there were long periods of silence, and otherwise to remain quiet, withholding information or advice.

Following the previously agreed–upon solution time (15 minutes in the first three sessions, 30 minutes in the last two), all participants assembled to report orally. We sat in a circle, and each observer reported on the emotions and actions witnessed during that observer's partner's problem–solving. Directly after each observer spoke, each solver could correct or add to the report.

The roles were then exchanged: observer became solver, and solver became observer. The new solver chose a problem, and the solving, observation, and reporting process was repeated. The data used in analysis consisted of the written notes from each observer, the problem solver's written work, written observations from the co–leaders, and the author's notes covering the whole–group discussions, observations, and discussions with the co–leaders.

RESULTS OF THE EXPERT PROBLEM SOLVERS

This section describes the emotional reactions that appeared to influence cognitive decisions during four particular stages in the problem–solving process.

1. The Initial Reading of the Problem
2. Embarking on the Problem-Solving
3. Immersion in the Problem
4. Solution to the Problem or near end of Session.

THE INITIAL READING OF THE PROBLEM

In three sessions, the expert problem-solvers were invited to choose their own problem from the books listed above. Their choice had to be made within a few minutes. As a result, problems were chosen that had at least some initial appeal to the solver. In the other sessions, problems were handed to the solvers. At no time did a solver refuse the problem, and solvers seemed to accept their problems with curiosity and positive anticipation. The initial reading of the problem provoked an immediate reaction based on identifying it by type followed by a sense of its difficulty. Two statements illustrate the initial evaluation of type and results. "I anticipate I will enjoy this problem but may not make much progress," and from another solver, "I loathe this type of problem. It is do–able but will require a big effort. I think I will have to go through many tedious decompositions."

The word "do–able" was used often and meant either that the problem was solvable or that progress could be made in understanding the question. As indicated in the second quote, the anticipation of being able to come to a solution of the problem and the enjoyment of working on the problem were not directly linked. All solvers were more encouraged by harder problems than by ones marked "obvious" or perceived as easy. Problems that were perceived as difficult were accorded greater value in the sense that intellectual effort must be expended to understand the problem. The intellectual effort was linked to the emotion of pleasure. Those who felt the problem worth working felt an immediate joy even before proceeding. This joy was a signal to bring all mental force to bear on the problem, which in itself produced pleasure and therefore motivation to continue.

EARLY IN THE PROBLEM–SOLVING PROCESS

The following episodes clearly show that emotions have a functional role in guiding the choice of method. After reading the problem, all solvers began to develop a notation, to draw a diagram, or to write out a hypothesis. They were totally focused on the problem at this stage. This was the beginning of a cycle of immersion in the problem discussed below. At this stage, considerable emotion accompanied the choice of method. In particular, solvers clearly expressed an emotional repugnance for any method which they considered "cheating" or "bad sport." Three definitions of "cheating" emerged. First, cheating, for these experts, meant any method or technique that was perceived as more powerful than the level of difficulty warranted by the problem. Each solver had already made a judgment about the level of difficulty of the problem based on the solver's perception of the type of problem, on how successful the solver had been with those types in the past, and on how much the solver enjoyed that type. For example, a university professor, Solver X, speaking aloud, said "Can I use fancy stuff? ... Then I'll use the Jordan Curve Theorem...." Then he laughed in an embarrassed manner, backtracked, and began to reread his notes, saying, "Maybe there is an easier way." Another professor, Solver Y, with a problem entitled, "An Obvious Maximization," spent several minutes resisting using calculus before finally making a grudging commitment to calculus techniques.

Using brute force was considered as bad as using a too powerful method. One solver expressed the feelings of several when, while tapping his pencil in an agitated manner, almost as if he were drumming, he muttered, "I'm annoyed because I cannot see any other way than brute force, and that would not yield for me any understanding of the problem ... there must be an easier way." Consideration of a too powerful method, brute force, or an "obvious method," brought forth comments and behaviors indicative of embarrassment or annoyance.

A less conscious resistance to cheating was seen when solvers imposed ridiculous restrictions on themselves. One solver, for example, had Honsberger's book in hand and was attempting to solve a problem that began, "Use the Method of Reflection to ..." The solver's reaction was, "I understand the problem but I don't know this method ... I wish I could read the chapter that discusses the Method of Reflection." The subject had self–imposed the restriction against reading the chapter. He considered it a form of cheating. Instead of simply turning to and reading the chapter, the subject tried to invent on the spot a plausible "Method of Reflection."

Another solver spent long moments in seemingly aimless thought, finally saying, "I'm feeling a little out of control of the problem ... There are lots of parameters ... There seems to be a lot of ways to define this problem ... I'd like to clarify the problem by asking whoever wrote it." The subject was holding back from defining the problem himself. Finally, with a forced air, the subject said, "I could break it up into cases myself and come to grips on my own terms and get partial solutions ... I've got control back."

In general, self–imposed restrictions or feelings of cheating would slow a solver down until he felt uncomfortable enough to say, "I'm wasting time. I really haven't done anything." Then there would be a squaring of the shoulders and a businesslike assertion to "... take a stand and try to prove it ...," even though this might mean grinding out a meaningless, albeit correct, solution. These excerpts from solvers' comments demonstrate the appreciation of an elegant, efficient, or otherwise aesthetically pleasing method of solution. While the desire to attain such a solution at times seemed quite conscious, subjects seemed less aware of the restrictions they imposed on themselves as a result of the desire.

IMMERSION IN THE PROBLEM

Once solvers made a commitment to a method of approach and made some progress, they became totally immersed in the problem. Their engagement was so complete that it appeared to make them almost oblivious to the observer, the environment, or themselves. At moments like this the solver would be writing on the paper or would appear lost in thought. Any verbalization to the observer was parsimonious. This level of intensity was not maintained constantly, but it was one mode of a cycle of attention to the problem (immersion), followed by attention outward to the observer or environment (distraction), followed by a diving back into the problem.

The distraction mode was prompted by several factors. When the solver paused overlong in appreciation of some success, then the solver's attention would gradually be drawn away from the problem to the environment or towards the solver's own self e.g., "I'm warm in here," or "It's warm in here." Likewise, the jolt of finding a counterexample to a hoped–for truth would prompt the solver to notice the ticking of the clock or the temperature of the room. To sit too long without progress would cause the solver to recount a memory of a similar past experience. Typical solvers, for example, after a lengthy period of frustration, would tell the observer that they were not really very good at this type of problem. Attention was diverted from a focus on the problem, to a focus on the self, sometimes accompanied by embarrassment. This outward attention was brief, usually less than a minute. Solvers would look around, stroke the pen, sigh, scratch, talk a little and then go back into the problem.

Although going too long without progress or action tended to bump the solver out of concentration on the problem, there were some activities that seemed to allow the solver to remain immersed in the problem even though no progress was being made. These activities could be called "breathers"; they were mathematical routines that required little cognitive energy. A typical routine was simply to rewrite the definition of the variable. One solver began, "There are two cases: (a) The problem is solvable, and (b) The problem is not solvable." Almost all solvers who began a problem using x's and y's, came to a point where they switched them to a's and b's, and vice versa. Another routine was to decide abruptly to use induction, and then write out the complete induction hypothesis. The problem–solving process took on the form of the following cycle.

(Attention On Problem) (Mathematical Routine/Ritual) (Attention Away From Problem)

The mathematical routine of switching variables or writing definitions appeared to provide moments of intellectual relaxation during which the solver continued to maintain an intense state of concentration on the problem.

EMOTION AT SOLUTION OR WHEN TIME WAS CALLED

Most of the solvers were in the immersion mode of concentration on the problem when it was time for the session to end. The emotion expressed by these people was irritation at being interrupted. Almost all of these solvers mumbled, "I'll continue later." They would have preferred to remain solving the problem. Those subjects who were in the distraction mode just prior to time being called, were usually aware that the time was just about up. They tended to sit back, observe others, and wait out the time. They did not work on the problem further, but mentioned that they would return to it at a later time. What became clear was that the attraction to the problem was balanced by the reluctance to allow oneself to get lost in a train of thought only to be yanked out of it.

The overall realization of the time constraint inhibited some problem solvers. An example of this was found in one solver's report, "I feel hemmed in. I do my best by playing around ... ordinarily I would draw pictures and really understand ... build up a pattern." He felt pressure to categorize a solution method quickly. "Without a time constraint I probably would have been more impulsive. I would have guessed and then worked backward. I felt forced to be more systematic, meticulous, and more step–by–step and mechanical. I think I could have solved this in a shorter amount of time if there had been no time limit." Although this solver produced a successful solution, the fact that there was a time–limit produced emotions that inhibited him from becoming as immersed in the problem as he would have liked. His attention was divided between the math–world and present time. Although he thought he could solve the problem in a shorter amount of time, he did not attempt to explore a shorter method. When the timing in itself counts, it is as though the meaning of the problem diminished.

Solvers expressed disappointment if the solution came so easily that little emotion was invested in the problem. Then they would remark, "The problem must have been too easy. I got it. So what's the big deal? I feel let down because I didn't spend a lot of emotion." When the complexity of a problem came like a revelation to one solver, he responded with a BIG smile.

One solver exhibited obvious arousal with eyes wide open, clear face and a slight laugh: "Hey, there's an infinite process ...." Exploration didn't bear out the infinite process, with the result of a clear drop of interest and a rather emotionless settling again into the problem.

EMOTIONAL RESPONSE TO USEFULNESS OF MATHEMATICS BY EXPERTS

A feeling of usefulness of mathematics played an especially important role for the women. Five of the eight women and only one of the 23 men brought up the issue of usefulness of mathematics. Subsequent to this study, one of the women has left her teaching career of twenty years to pursue a degree in another field in which she feels she will be able to make more of a contribution in the world. A different participant clearly describes the strength and influence of feelings of usefulness. This participant was a medical doctor with a substantial background in mathematics. She was able to solve the assigned problem in a short time and with no intense engagement. She was disappointed and felt let down. She insisted her feelings would have been the same had the problem been more challenging. In a loud and agitated manner she described her feelings:

> What would have been a meaningful problem? How come I'm not satisfied? I had an expectation about solving that problem that did not get fulfilled. It didn't make me happy. There were some moments of tension and some of excitement, but not intense. It was entertaining like a Grade C movie.

She had related the value of the problem to the intensity of emotion involved in its solution. She went on to describe the usefulness of mathematics:

> Math has no social relevance to me. I am willing to solve math problems, even ready. But it feels completely disjoint from what interests me. I still love it, but its importance seems minuscule compared to world problems ... beautiful but frivolous to use my mind in this way.

She said that, while it might be useful for other people to do mathematics, there were more pressing issues for her to engage in. Mathematics has an esthetic appeal for this solver. She sees mathematics as beautiful and is attracted to mathematics. She is confident. She was successful in reaching a solution and feels ready and willing to solve others. It seems apparent that the intensity of her medical problem–solving and its immediate applicability have set a threshold of "intellectual utility" that recreational mathematics cannot reach. However, this episode and the comments of the other women show that the emotion attached to

usefulness of mathematics can be overriding and can serve as a powerful influence on whether or not to engage in mathematical activity at all. This is in accord with research by Fennema (1989) and others who have shown perceived usefulness of mathematics to be one of the most important variables in gender-difference research.

SUMMARY OF THE EXPERT GROUPS

Emotions influenced the problem–solving process both positively and negatively. Excitement, hope, eagerness, and the "joy of battle" were positive emotions mentioned by the solvers. The positive emotion of hope provided motivation to keep going. Occasionally during problem–solving, the solver lost control of the problem. One solver said, "This is too complicated, too many angles to label," and another, "I feel this is getting a little out of hand. This one and that one cancel out and I haven't used the fact that it's a prime." The solvers had confidence in their own abilities and experience with doing mathematics, so they did not lose hope and did not stop working. Hope, the belief that there is even a slim chance things will work out, helps one continue. Ambiguity nourishes hope. One cannot be hopeful when the outcome is certain.

Solvers who perceived their problem as too easy felt disappointment even before they began to work on the problem. The emotions of those who perceived the problem as a challenge are summarized by one subject as "the joy of mental engagement and the bringing of all mental force to bear in a cohesive way." The mental engagement took on the form of total immersion in the problem. There was a Lorelei seductiveness about it, a delicious slipping off into another world. During discussions after the sessions, people revealed times when they used mathematics as an escape from daily life, such as to ignore the pain of an illness or to avoid thinking about someone. Mathematics can help with depression, as the famous mathematician Kovalevskaya wrote, "I am too depressed ... in such moments, mathematics comes in handy, and one enjoys the existence of a world completely outside of oneself" (Koblitz, 1983).

Mingled with the charm of seduction, the math world takes on a dangerous quality, a frightening isolation if one stays immersed too long. Rosamond (1982) has given examples in which problem-solvers felt consumed by a too dominating mathematics. As one distraught mathematics graduate student said with tears in his eyes, "What do you do if you are 80% — 90% mathematics? If you've let yourself become consumed by mathematics so that that is what you are and then you want to let someone get to know you? What do you do when you can't explain that much of yourself to them?" Working with an observer was a protection against becoming dangerously immersed or overly isolated. We wonder how professional mathematicians relate to these faces of mathematical life.

EMOTION IN NOVICE PROBLEM–SOLVING

The second study involved mathematical novices, consisting of six students in an intermediate algebra course, 14 computer science and two mathematics majors in a college algebra and trigonometry course, and 12 mathematics majors in two history of mathematics courses, all at a private university. The study consisted of six activities. In all the activities the students chose their own partners and worked in pairs. The two initial exercises were short discussions of only two minutes' duration. The second two were three–minute visualization exercises. The last two were problem–solving sessions of about ten minutes each. During each activity, one partner was a silent, attentive observer who took notes on the partner. Then the entire group came together and each observer reported. Each problem-solver could amend or add to the report. The observer/solver then switched roles in each pair, and, after the second person worked on a problem, the entire group again gathered together. The data for the research include the notes taken by each observer, the work done by each solver, and notes and tape recordings of the group sessions made by the researcher.

Whereas the format of the exercise was similar to that of the experts, the experts gathered together for the fun of the investigation and activity, while the novices were students compelled to do whatever the teacher suggested.

Having the students discuss the exercise together as a group was an especially useful format. Students would add to and build on other students' comments. A comment by one student would remind another of some idea. This is similar to the experience Casserly reported when she interviewed students in groups while studying feelings of gifted students enrolled in advanced placement programs (1979).

RESULTS FROM THE NOVICES

The following data was compiled from the observation reports of the paired problem–solvers, observations of the teacher, and audio tapes and notes from the discussions following the problem–solving

sessions. While conclusions have been drawn, these must be regarded as tentative, intended to be provocative of further research. The results will be described briefly and compared with the data from the expert solvers.

INITIAL EXERCISE

The initial discussion activity was chosen with two purposes: to provide students with practice in paying attention to a peer, an important experience for an observer, and to provide practice in describing feelings about mathematics to a peer, an important experience for a problem-solver in this research. The initial activity, repeated on two occasions, was for student pairs to spend two minutes each, one person talking and the other giving attention, discussing a statement provided by the teacher. The first statement was, "Why I like mathematics," and the second statement was, "Why I am an excellent problem solver." In the general discussion that followed, several students agreed with the algebra student who said, "At first I said that I wasn't a very good problem solver. But then, I still had more time and so I began to think of ways in which I was." An all A–student who is very good in mathematics said, "I felt like I was my own cheerleader. It felt good."

In the brief paragraphs students wrote about the activity, several students expressed a sense of relaxation. For example, "The short session helped to relax me a little. It was enjoyable to talk for a few minutes and relax. It kind of helped my mind to relax, and then later some items that I found hard to understand kind of came together all by themselves."

Another student wrote, "I feel it (the activity) creates a positive atmosphere and provides a short escape while not totally losing focus on mathematics. I liked it."

Most students found the activity unique in that this was the first time they had ever thought about their feelings with respect to mathematics. An algebra student wrote, for example, "I felt kind of funny doing tonight's exercise, but its a good idea to bring out the positive points of math and it surprised me how many things I like about math."

Another student made a similar statement, "I feel that it released feelings I have about mathematics by saying them out loud for the first time. Also, by listening to another person's thoughts on why they find mathematics interesting. At first I felt stupid, but now I can see ... some more potential in doing this ...."

A computer science student wrote, "I had never verbally given my reasons and feelings about mathematics to anyone. This is important to get a real, solid reason for being here, because other curriculums would be less demanding. I know from experience that you must set out your goals to develop commitment — and this exercise starts me along the process of developing strong commitment." Another student seemed to second this opinion by writing, "I believe this is a good idea because if a person does not feel good about himself then he is not going anywhere no matter how much he learns."

A mathematics major summed up the general positive reaction to the activity by writing, "This (activity) is a great idea. We are not always willing to open up to each other but ... why not? Working on feeling good about yourself can never be bad!!"

Some students were more enthusiastic than others, but every student was intrigued and willing to do the activity, including the few for whom English is not their first language and who expressed some difficulty with communication. The activities provided a relaxed way to begin to discuss feelings. It was deliberately kept short so as not to intrude on other classroom work.

VISUALIZATION AND TEXTBOOK PROBLEMS

The last sessions were preceded by a short, teacher–led discussion of feelings and mathematics. The teacher pointed out that feelings are difficult to describe and that sometimes it can seem as if there are not enough adequate words. She asked the class to provide words and phrases that expressed their feelings about mathematics. The teacher listed these on the board as quickly as the students called them out. The list of words and phrases can be found in the appendix.

In the third and fourth sessions, a problem was provided by the teacher with the instructions that no pencil-work was allowed for at least three minutes. That is, the solver was instructed to visualize the problem and perform stretches, rotations, and other transformations mentally. Students relaxed with eyes closed while the teacher read the problem aloud. They were invited to ask questions for clarification. One person in each pair described thoughts and feelings to a partner. The two visualization problems are described in the appendix. Students tended to find this activity frustrating, and it did not appear to enhance the subsequent pencil-and-paper solutions.

For the last two sessions, each solver chose either a homework–type problem from a textbook or a problem from a puzzle-book. The time allocated for working on the problem was about ten minutes, followed by the group's gathering to discuss the observations. Then the roles of solver and observer were exchanged and the process repeated. The discussion of the novices will largely parallel that given above for the experts.

COMPARISON OF NOVICES AND EXPERTS

Some of the emotional aspects of problem–solving exhibited by the experts were observed in the novices; however, it was clear that the novices had much less mathematical experience on which to draw, and more recent negative experiences.

THE INITIAL READING OF THE PROBLEM

The students in the college algebra and trigonometry course were given their choice of three problems from their textbook to solve. Reasons for choosing a particular problem varied, from comments such as, "It seemed like something we had been working on, familiar," and "It makes sense and I can draw the starting picture," or "It looks doable," to "Basically, because it is challenging." One student chose a clock problem because she did not want an airplane problem; another liked aviation. These students had some sense of the type of problem they chose, and they tried to judge the amount of effort that would be required. For most students, the problems turned out to be more difficult than they expected. Initial confidence was replaced by frustration at being stuck, anxiety over not being the first finished, confusion because an apparent answer didn't look right, relief upon realizing some algebraic miscalculation or relationship error, and a sense of satisfaction or letdown when time was called.

The variety of feelings as described in these comments written by one observer are typical.

O (Observer): Why did you pick this problem?

S (Solver): It was most recently covered in class.

O: How do you feel about math?

S: I have no strong feelings about math.

He reads the problem.

S: Oh, man! I'm no good at these!

He draws a picture for the problem, then says, "That's easy." He explains the variables, puts them into the Law of Sines formula, and starts to solve.

S: I feel very happy now, I've got it! ... I'll check.

He rereads the problem, then loses confidence in his answer.

S: I'm not sure if I solved for the right thing.

He looks, re–reads, re–draws, tries many re–draws, says, "Oh, I've got it!" and attacks his calculator with gusto. He works out his new idea and solves the problem.

S: I feel pretty good about this answer.

He says, "It is very satisfying to solve a math problem." and gives an arm pump.

On the same paper the problem solver has added, "I didn't pay too much attention to the observer. It was rather motivating for me to be observed."

One of the observers of a mathematics major reported that her partner realized the problem was going to be tedious. "Not that he couldn't find an answer, but just that it would take a lot of time and a lot of brain power. He seemed impatient, as though, 'I don't want to have to sit here and have to do this and take all the time to do it.'"

As with the experts, the novices had some evaluation of the type of mathematics problem they had chosen. After reading the problem, they anticipated pleasure or frustration, and they had a sense of the effort needed for results. Whereas the students' limited experience meant they had fewer methods of solution available to them, they still made an emotional appeal to a sense of appropriateness.

The mathematics majors were offered puzzle-books and textbooks from which to solve problems. Ten of the twelve chose a problem from a puzzle-book. One student explained, "I like word problems ... and it's a real world type and I like that sort of thing." She began the problem and then said, "I changed variables. I changed the way I put my equations. I thought I'd try simultaneous equations. Then, I said let's try some numbers. The minimum it can be is this ...."

T (Teacher): When you switched variables, did sometimes you use a's and b's and other times x's and y's? Did you switch?

S (Solver): Yes.

Almost every problem-solver reported changing variables or re–drawing a picture during the solving episode.

> T: Sometimes people don't want to use a more high powered method than the problem deserves. Is that what was going on with you when you said first you were going to try simultaneous equations and then you seemed to back off and decide to just try some numbers?
>
> S: Yes, it shouldn't be that difficult. It (the problem) sounded easier than this so I shouldn't have to use something like that.

In one case an almost uncontrollable negative reaction during the initial reading of her problem severely inhibited an algebra student. She reread the problem repeatedly both silently and aloud, but did not appear to be thinking about the words. She admitted being anxious. "When I look at a math problem that seems really different from one I've just studied in the book, I get so anxious. I want to solve the problem but I don't know how to start. I feel bad and then I feel bad about feeling bad." This student spent much of her time telling the observer about how she could understand a problem when it was explained in class, but how she would struggle unsuccessfully alone at home. Her anxiety reaction is not an uncommon one. Turner (1988) describes the rehearsed pattern of failure that becomes "not only well established and familiar, but an acceptable, and ultimately relied upon, event." Just as the confidence expressed by the experts provided them with a positive anticipation toward reading the problem, the lack of confidence expressed by the algebra student blocked her from even trying to make sense of the problem. She is reacting as Mandler (1989, p. 244) has suggested, with autonomic arousal affecting her access and retrieval of knowledge stored in memory. "The occurrence of strong negative affect produces an immediate attempt to remove the reason (the cause)." The caring attitude of the observer made a difference for this student. (All partners stayed together for all activities.) Her partner gently reminded her that she had excellent problem–solving skills, and he knew this because she had told him. After this small positive affirmation from her partner, the student read the problem through once more, very slowly, drew an accurate picture, and mumbled as she worked, "Yes, I have excellent problem–solving skills." She determined that the triangle she had drawn was too large but was unable to continue because the time was called. What is particularly interesting is how quickly the intervention of the observer influenced the problem–solver's ability to make progress on the problem. The role of the observer will be discussed further below.

Immersion in the Problem

Each of the mathematics majors reported being focused on the problem to the exclusion of anything else going on around them. They described the modes of immersion and distraction. Student B reporting on Student T says, "He did not want to stop when time was called. Basically, he was so intrigued with the problem that I was working on that he just went right back into it ... he was enthusiastic and using new variables and talking through the whole thing ... he was concentrating and very methodical and checking things off, working and checking, and he kept this up for some time, and then paused, rubbed his chin and said, 'I'm tired of this. I'm going to try something different.'"

"He was looking for new relationships and the time went by very quickly. I don't think he noticed that anything else was going on in the room. And I felt the same way working on the problem. You have no concept of anything else going on. It is real focused. Anyway, he was determined to keep going, he didn't want to quit (after time was called). He still had a lot of energy left."

> Student T: What is it that bumps you out of that concentration state?
>
> Student B: For me it might be someone's real loud voice. It's something startling that will get me out of it ...
>
> Student T: What is it that gets you out of it when nothing startling is going on and you are doing the mathematics? What's going on in your problem–solving that gets you out of it?
>
> Student B: Solving the problem or getting to a point where I really don't know where to go next, and then I just have to get up and walk away. But we didn't have enough time for me to get to that point yet because there was still energy left for this problem.
>
> Student T: Yes, I agree. Either solving the problem or you are just going in circles and so your mind wanders, you are thinking of different ways to approach the problem and then your mind wanders and you think of the beach ... Your mind

> wanders in search of a new idea and it happens to take your attention away from the problem in the course of wandering.

Negative self-talk also can take attention from problem–solving, as reported by Student W, a mathematics major.

> Student W: If I'm concentrating on a problem and I get to a point where I'm so frustrated, then the thoughts begin to run through my mind: Look at this, you can't do it. How stupid. It's ridiculous that you cannot do this problem. Why can't you remember? Gosh, anyone can do this, you moron. There is a lot of negative talking to myself. That's when my thought is that I just got to give up. It's just a feeling. It's not giving up forever. It's just a feeling of giving up until I can regroup and come back to it. But for now, forget it.
>
> Student T: How long does this last? For minutes or for days?
>
> Student W: Ahhh, just at the point that I feel it. Once I put the pencil down and go do something else, I don't feel it anymore. I know that when I come back to it, if it's something I've done before then I'll remember it. If it's something I've never done before then, then I may not get it at all.

There was more dialogue between the novice solvers and observers during the problem–solving sessions than there had been with the experts, and observers had more difficulty restraining themselves from giving hints or explaining what they would do. One solver reported being angry that his partner wouldn't help him. He was convinced that together they could have solved the problem. Observers would remind the solver of a problem–solving heuristic now and again (some of the classes had compiled a list from those of Pólya, Brown, Schoenfeld, and others) or ask leading questions.

Solution of the Problem by Novices or End of Session

Most students expressed irritation when time was called and said (as did the experts) that they would continue later. Student J had solved one problem and was in the process of working on a second.

> Student J: I was still coming up with a way to solve as opposed to an actual solution ... There was some impatience because the method wasn't jumping out at me, but beyond that it wasn't a real emotional experience.
>
> Teacher: Was the first one, where you solved it, a bland experience?
>
> Student J: Actually, when I read the problem, to be honest, I felt pleasure because it was so obviously clear that it was solvable quickly. There were not many variables. It just wasn't that complicated. Although I didn't know what the immediate answer was, I felt like I could solve this one right away and there was pleasure in knowing that. Then I went ahead and solved it and I got confirmation of that because I did solve it.
>
> Student M: When I finished mine I thought, "Oh, I can't believe I did it. It can't be right." I looked at the answer thinking, It can't be that easy.
>
> Teacher: When you finished that problem, did you have a sense of accomplishment, puzzlement, or what?
>
> Student M: Puzzlement — accomplishment too, but "Did I do it right?" I figured out the problem according to the parameters I made, but I didn't know if they were what was wanted.
>
> Student R: Some people have said they feel a sense of let–down–ness if they solve a problem that is too easy. Does that make sense to anybody?
>
> Student W: If it doesn't seem easy at first, if when I first read it I say, "Hmmm, am I going to be able to solve this?", then if I get the right answer, but if it takes me only five minutes to do, then that feels good. But if when I read it it is just a word problem, I can do addition of apples and oranges, then that's not hard at all, then that's not ...

Student B: If there's not a little bit of challenge, then there's ... not a point of doing it ... the challenge of overcoming that ...

Student M: For me, there's a big let down. You get into the problem, getting all ready, your brain is mustering its all to throw at this problem ... Wait a minute! I could have done it at half-strength! I didn't need everything on this one.

As with the experts, the emotional charge for the novices appeared in direct proportion to the perceived difficulty, except when the problem was seen to afford a quick, simple, solution.

EFFECTS OF THE RESEARCH PROJECT ON SUBSEQUENT STUDENT BEHAVIOR

Conversations with students after the course showed that the research project had introduced students to a new way of working problems which appeared to have a substantial effect. It was clear in the class that the pairs began naturally to turn to each other to work problems together. They also talked to each other about their math outside of class. Students reported that, when they were working alone on a problem, they could imagine their partner with them and what the partner would be saying. When I (the author) asked what this would be, the response was that the partner would be supportive with statements such as, "Come on, you can do it," and inquisitive with questions such as, "Where are you stuck? What part of the problem are you on? Convince me you are right." Students said that this calmed them down and helped them back away from their confusion and look at the problem afresh. (In class, we had spent some time working problems and identifying the "Phases of Work" described by Mason, Burton, and Stacey [1982, 1984], and using the "What if Not?" techniques of Brown (1982).) These responses confirm the results of Clement and Konold [1989].) who found that "having students work in pairs alternating between the roles of solver and listener helped students internalize the roles of solver/listener and learn to carry on an internal and critical dialogue with themselves." Students in this study reported that they felt a heightened responsibility to be able to explain, and verify their solutions.

In summary, paired problem-solving begun in the classroom was continued by the students outside of class. Students wanted to come to class in order to see their partner or to support their partner. Students began to internalize the roles of solver/observer, to "hear" the positive affirmations of their partner, and to ask themselves the questions they knew their partner would ask.

EMOTIONS, BELIEFS, AND THE ROLE OF THE CLASSROOM TEACHER

Mathematics can be considered as a way of viewing the world, and its meaning and process are subjective and individual. On the other hand, mathematics is a common language with rules and precision. From Cobb (1989) we have, "The complementarity that seems endemic to mathematics education theorizing expresses the apparent paradox between mathematics as a personal, subjective construction and as mind-independent, objective truth." Many students tend to see mathematics as precise, rule-driven, objective truth rather than constructible knowledge. The terror that comes over some students when faced with a mathematics problem can be related to their belief that the problem is an all-or-nothing situation, "I either know IT, or I don't, and there is no use in my trying to figure IT out" (Copes, 1982, p. 24). In addition, students at early stages of intellectual development with respect to the discipline will ascribe to the teacher the sole Authority and repository of the Truth (Copes, 1991; Perry, 1970).

Furthermore, whereas the goal is to assist the students in the construction of their own mathematical knowledge, teachers still tend to feel fully responsible for the mathematical learning of their students. Fischer (1988) describes mathematics teachers as tending to "identify themselves with the subject matter and present mathematics as if they were totally convinced of its truth and worth." Thus, there is a sort of collusion going on in classrooms between students who view the teacher as the Authority and sole distributor of mathematical Truth, and the teacher, who has assumed the embodiment of mathematics.

Paired problem-solving can help disengage the collusion. Research by Perry, Copes, and others on the Perry scheme of cognitive and ethical development and on Kohlberg's model of moral development shows that students did not move forward in development when they were in an environment more than one stage later than their own, but did move forward when they were in an environment just one step beyond their current stage. Pairing students puts them with peers that are usually at the same or slightly later stage, and thus can be a force toward cognitive and ethical growth.

Other instructional environments and teaching techniques help remove the burden of full responsibility away from the teacher. Carefully designed computer labs such as the one designed by Dubinsky (1988) and Schwingendorf at Purdue propel the students into relying more heavily on themselves, the computers, and their lab-mates. Teaching the students to read mathematical textbooks and literature, as Clarence Stephens has done at Potsdam, provides the students with an "Authority" other than the teacher.

Along these lines a wide variety of successful programs was described at the first National Conference on Women in Mathematics and the Sciences (Keith and Phillip, 1989).

PHENOMENA COMMON TO BOTH EXPERTS AND NOVICES

EMOTIONS AND THE INFLUENCE OF THE OBSERVER

Contrary to almost everyone's expectation, being observed while working on the mathematics problem was a positive experience. Both experts and novices reported that being observed, "Made me feel important" or "... that what I was doing was important." An expert reported, "I felt honored that another person was taking the time to observe me." Another expert said that the analytic intimacy she felt while being observed was one of the most poignant mathematical experiences she has ever had. "It felt intimate to have someone committed to watch the inner workings of my mind." Even those who felt frustration with the mathematics problem enjoyed the process of being observed.

An expert related the process to the testing situation. "A test is an almost random set of narrow problems where one thing must trigger another. It is not about figuring things out. Test questions do not show that math is a process." This expert did not solve the problem, but was not intimidated by being observed. "The observer could hear that I have math training. He could see how my math mind works, how I assimilate information, manipulate, and use an arsenal of strategies. This is so much different from taking a math test where I am not tested on how my mind works. On a math test, I could expect not to be able to show what I know. I would feel shame."

One of the mathematics majors commented, "To be perfectly honest, there is a thrill to the risk because if you are unsuccessful you are damned . So there is a certain amount of thrill that goes with the risk and so it becomes very positive when it reaches a point in problem–solving when I can see how to solve the problem and I can say, 'Oh, accolades are coming.' That part of the thrill becomes positive. And, of course, there is the other side, and I've experienced it often, and that is, 'Oh, I'm not going to get it in time!!' And know I have the potential of being sent to the corner over there ... but there is that risk and the elation that comes with risk."

Most of the solvers continued working on the problem longer than they would have, had they not been being observed. One novice explained, "I felt I owed it to my observer to do my best." Almost all claimed to like their problem more the longer they worked on it, and those who did not like the problem initially began to like it and to get interested in it. Without an observer, these solvers might have quit.

THE EFFECT OF REPORTING IN A GROUP: EXPERTS AND NOVICES

The group-reporting session provided an effective mitigation of math anxiety for some solvers. An expert, a math professor, revealed, "The most moving part of the exercise was hearing the observer say what I'd done. I did not feel intimidated. I didn't get any of the bad response I expected. The observer demystified my emotional and intellectual engagement by simply listing what I did: 1, 2, 3, 4. This cut it down to size, gave it true proportion." The emotions felt by this solver, her "private agony," were reported in a matter–of–fact tone by the observer. The solver was surprised; and later said her reaction was, "Is that all? Is that all I did? Well, that wasn't so much." Hearing her emotions described flattened out their intensity.

The observers found that the process of observing appeared to diminish some of the secret charge of their own anxieties. "I could recognize my feelings in the other person, and could see how the feelings influenced what he did," reported one observer. "I kept thinking, 'Why is he spending all this energy fussing about it? Why doesn't he just get on with it?'" Observing the emotional energies of another person helped observers to evaluate their own.

A mathematics major provided another positive benefit to the group-reporting process. "When we are working the problems, we are doing stuff that we don't realize we are doing. My observer reported that I was being methodical, and I was. But I didn't really realize I was doing that. And someone else was talking out loud and didn't realize that they were. I don't know if it is beneficial or not, when you don't know and then someone is watching and tells you, then you do know. You see what you are doing that you didn't know you were doing."

> Student B: You learn that other people feel the same things as you so it can build a relationship within the class. You also learn how to approach problems, for example, almost everyone in this class read the problem at least four times.

Student M: Other people's emotions are usually a good mirror for oneself. What are they looking like? What kind of body language are they sending?

Student J: One of the things that is coming quite clear is that this whole process really has quite an emotive aspect to it. It's not quite as detached or unattached as I had previously thought it was.

One solver reported that being observed enhanced his precision. He felt he wanted to perform well. Both experts and novices reported an initial reluctance to free–associate ideas in front of an observer who might have already mentally solved the problem, or who might be bored, and some solvers wanted to talk the problem over with the observer, or would look up at the observer hoping for confirmation. All solvers reported an increase in interest in the problem and in solving it due to the presence of the observer and due to the act of reporting.

CONCLUSION

The research appeared to show that emotional elements interact with cognitive processes and exert strong influences during mathematical problem–solving. Solvers anticipated leisure or frustration upon first reading of the problem. They anticipated the amount of effort that progress or results would require. This evaluation and anticipation helped determine the choice of method. Solvers' (especially experts') concern with not using a too–powerful method sometimes caused them to put too many restrictions on themselves. Solvers alternated between immersion and distraction during solving, with emotional satisfaction relating directly to problem difficulty.

The presence of an observer during problem–solving was a highly pleasurable experience that appeared to support persistence, sense of self–worth, and the development of a cognitive and affective monitor.

The process of coming together for the observers to report to the entire group and to hold group discussions helped students see the emotive components of what they were doing. This exercise might be especially useful with future teachers, who may need help in recognizing some of the positive and negative student emotions.

The results may be helpful for helping the teacher design an environment to accommodate student diversity and promote cognitive and ethical growth. Future research may examine the subtle interaction between cognitive processes and specific emotional constructs such as value, motivation, hope, or, as Silver and Metzger (1989) have done, aesthetic values.

APPENDIX I
TWO VISUALIZATION PROBLEMS

1. Imagine a belt around the equator of the earth. The belt follows the contours of the earth and is snug. Now, imagine adding 40 feet to the length of the belt. Evenly distribute this added 40 feet all around the earth. First the belt was snug, now it is loose. What can fit between the belt and the earth?

2. Locate the point A(3, –3) on the coordinate axes. Draw a line through this point, making a triangle with the coordinate axes of area 6 square units.

APPENDIX II
LIST OF FEELING DESCRIPTORS DEVELOPED BY NOVICES

I had introduced this activity by saying that some people liked mathematics because it seemed clean, with only one right answer, unlike poetry or literature, which is open to interpretation. Student B, a mathematics major, said that, while it is true that she used to think that mathematics had all the right answers, and she had confidence she could find them, now, as she has gone on in mathematics, it seems to her to be more and more like poetry, a sense that it is aesthetically pleasing.

Student J described knowing that everything he was doing was dove–tailing to a final answer, even though he didn't know what it would be. He knows and doesn't know simultaneously.

> Student M: When I work hard on a problem, I feel mentally taxed, like I have been exercising my brain. I need a mental break.

Students called out these words or phrases as quickly as I could write them on the board.

aesthetically pleasing
simultaneously knowing and not knowing
mental exercise

satisfaction	concentration	right/wrong
frustration	exaltation	anger
focused	disappointed	confused
excited	industrious	thrilled
accomplishment	hesitation	validation

I mentioned a "lull period" during which a solver waited to find what thoughts would come next. Students identified with this and suggested: incomplete, static, and blocked. One student described it as "Working a process and the process is working and you know what comes next. You are moving from point A to point B." Another student added, "A sense of growth, anticipation, relief." Then I mentioned "isolation" and another said "a feeling that everything is sucked into the problem–solving process."

ROSAMOND — COMMENTS

This study, labeled as preliminary, probes ground that is new in the sense of being newly subject to systematic inquiry, but is as old as human experience itself. We all know emotion when we experience it, and we all can recognize it in the outward demonstrations of another. And somehow, even that most cognitive of activities, mathematical problem–solving, is deeply enmeshed with emotions of many types and hues. While one might (emotively) yearn for certainty and precision in research, sometimes it is impossible and inappropriate. Fine–grained statistical data on attitudes towards mathematics, on confidence or anxiety, or even on aesthetic judgment of some piece of mathematics based on survey or other data-gathering instruments, may not be quite appropriate in this case, where attention is on the role of emotion in the *process* of problem–solving and where the phenomena to be studied require further identification and clarification.

The author and colleagues examined emotion occurring during paired problem–solving sessions involving several dozen problem-solvers — about 30 experts, ranging from reasonably mathematically-experienced teachers and users of mathematics to professional mathematicians at the faculty level — and a like number of novices, including undergraduates in low and intermediate math courses, as well as a dozen math majors. Data on experts was based on self–reports (concurrent and retrospective) as well as observer notes taken by both the observer–half of the problem–solving pair and the author or her colleagues. Novice data is described later.

The role of aesthetic considerations seems to be central among experts. Is aesthetic judgment behind emotional response, or is it the other way around? How are they related? Solvers seemed quite tightly constrained by aesthetic judgments of the match between solution method and problem level, and by their perception of economic and elegant solutions versus. labored case analyses, for example.

An interesting cyclical variation in attention appeared during problem–solving, where the solver would alternate between intense concentration on the problem, superficial or routine activity (e.g., rewriting the problem statement, cleaning up equations, et cetera), and distracted attention to the external environment. Rosamond also found that imposing time limitations seemed to devalue the problem–solving strategies — solvers felt constrained to be more systematic and less exploratory than they preferred, and less invested in the particular methods they used. Interestingly, five of eight women, but only one of 23 men mentioned the usefulness of mathematics in their discussion. One of the women, a physician, found mathematical problem–solving very unrewarding, almost frivolous, because of its seeming inutility, despite her considerable ability. Perhaps her judgment and feeling were based on a comparison with the problems she solved in her daily work.

Data gathering with the 32 undergraduates was more structured — alternating paired problem–solving and taped group–debriefing discussion sessions. While obviously these students had fewer intellectual resources at their disposal, and the non–mathematics majors among them showed considerable fear, anxiety, and tendency to frustration, they shared some of the same aesthetically based affect that was seen among the experts. The mathematics majors especially showed some of the same cyclical behavior, and follow–up interviews after a course that emphasized paired problem–solving revealed the same consistent tendency that others have reported for the students to internalize the monitoring and reflective role of the partner, which positively affected their own problem–solving behavior when working alone.

The bulk of this paper amounts to identifying anecdotally phenomena that are worthy of subsequent inquiry of a more systematic, although not necessarily quantitative nature. One would hope to see more definitively operationalized characterizations of the cyclical attention phenomena, for example, as well as further exploration of relations between aesthetic and affective factors during the problem–solving process. But we see as essential the focus of such inquiry on the processes themselves, especially as the processes themselves vary — across populations and across problems. It seems that only through such variation are we to have a chance at identifying underlying invariants, especially those that take the form of critical cognitive/affective relationships.